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ASYMPTOTIC BEHAVIOUR OF IMPERFECT VISCOELASTIC BEAMS

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Abstract—In this paper the problem of the lateral stability of imperfect viscoelastic beams is analysed. The problem is examined by means of the quasi-static approach and leads to a system of integro-differential equations which can be resolved by a series expansions and the Laplace transforms. The viscous critical load is defined according to the asymptotic behaviour of the beam and can be evaluated by introducing the assumption of weak fading memory, solid material and thermodynamic compatibility. The critical load does not depend either on the type or the entity of the imperfections. Furthermore, some characteristic aspects of the problem, like the faster progress of the torque moment with respect to the progress of the lateral bending moment, are underlined. For a three-element model it is possible to reach a closed-form solution. In the case of the Poisson ratio constant in time and three-element model, an interesting analogy with the asymptotic behaviour of the imperfect column is observed.

NOMENCLATURE

a, b, c, s_0, s_1, s_2	parameters for describing the solution
$E(t), G(t), e(t), g(t)$	functions for describing the relaxation
E_0, G_0	axial and shear elastic modulus
E^*, η, G^*, γ	constants of the three-element model
$\varepsilon_{xx}, \varepsilon_{xy}, \varepsilon_{xz}$	components of the strain tensor
$\phi(x,t)$	rotation of the cross-section
$\phi_0(x)$	initial imperfections given by rotations
J_t	torsion modulus
J_z	momentum of inertia with respect to z-axis
M_t, M_y, M_z	moments with respect to x, y, z-axes
m_i	ratio between the bending moment and the ith Euler bending moment
ν	Poisson ratio
$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}$	components of the stress tensor
p	ratio between the axial load and Euler axial load
$v(x,t)$	lateral displacements along the y-axis
$v_0(x)$	initial imperfections given by lateral displacements.

1. INTRODUCTION

The stability of viscoelastic structures is always of particular interest because the external load giving rise to structural failure over a long period is notably less than the critical load obtained by an elastic analysis at the initial instant. The judgement on the stability of a motion for a viscoelastic system is usually defined on the basis of the features of the motion caused by a small variation of the data (disturbance). If the motion consequent to the disturbance is bound then it is stable, otherwise it is unstable. In the past, the most meaningful analyses involving viscoelastic stability dealt with the column problem underlining some important questions (Dost and Glockner, 1985; Szyszkowski and Glockner, 1985; Russo Spena and Sparacio, 1989), whereas works on the problem of the lateral stability of deep beams are not available in literature.

In this work, the authors intend to deal with the latter problem by studying the behaviour of a beam subjected to a constant bending action and constrained at the ends, assuming the small deformation theory and a linear isotropic viscoelastic material. The disturbance consists of geometric imperfections and the inertia forces are neglected (quasi-static approach). In contrast to the column case, the bending action and the torque moment act together to produce a biaxial stress in the material, therefore two different temporal functions are required to describe the constitutive law of the material.

The examined problem is governed by a system of two integro-differential equations which lead to a well-posed problem (Volterra equation) for a bending moment smaller than the Euler moment, consequently a bound solution is obtained for every bound time interval. Therefore, as already noted in the case of the column, a finite critical time cannot be defined in this case either and the instability conditions can be reached only in infinite time, as shown by Hilton (1961) and Kempner and Pohle (1953); the critical load is then defined with regards to the asymptotic behaviour.

The problem has been approached by means of series expansion and Laplace transforms. By introducing the assumptions of weak fading memory, solid material and thermodynamic compatibility, investigated by Fabrizio and Morro (1988) and Fabrizio and Lazzari (1991) and Giorgi (1989), it is possible to achieve a general stability condition and to prove that a viscous critical load, depending only on the viscoelastic parameters, can be defined. This critical load is smaller than the Euler load and, in the case of decreasing relaxation functions, can be directly related to the asymptotic moduli of the material. With regard to the evolution in time of the phenomenon, it is observed that the progression of the rotations is always more rapid than the progression of the lateral displacements; it follows that the torsional action grows more rapidly with respect to the bending moment. In this stability problem, the imperfections may be introduced in two different ways: through an initial displacement or an initial rotation of the cross-section. The problem has been approached in both ways, showing that the asymptotic behaviour is the same and that the viscous critical load obtained depends neither on the entity nor on the type of imperfections, even though the solutions progress in time in a different manner.

The previous qualitative analysis has been integrated by performing a closed-form solution for a simple viscoelastic model as for the three-element model. In this case it has been possible to compare these results with those of the better known problem of the bending stability of an imperfect column where the viscous phenomena considerably affect reliability. In the particular case of the Poisson ratio constant in time, an interesting result is obtained: the ratio between the viscous critical load and the Euler load assumes the same value in both cases. Therefore, even in the problem of the lateral stability of deep beams, viscosity notably affects reliability.

Some aspects of the problem are discussed in a numerical example.

2. STATEMENT OF THE PROBLEM

2.1. *Constitutive equations*

It is assumed that the material is isotropic and linearly viscoelastic. The isotropy assumption permits the constitutive law to be expressed simply by two independent temporal functions; in the examined problem, it is useful to define two functions that make it possible to simplify the formulation by leading to relationships which are formally similar to those obtained in the elastic De Saint Venant theory of rods. In this theory, it is assumed that the stress components σ_{yy}, σ_{zz} and σ_{yz} are null (see Fig. 1), and that it is possible to express

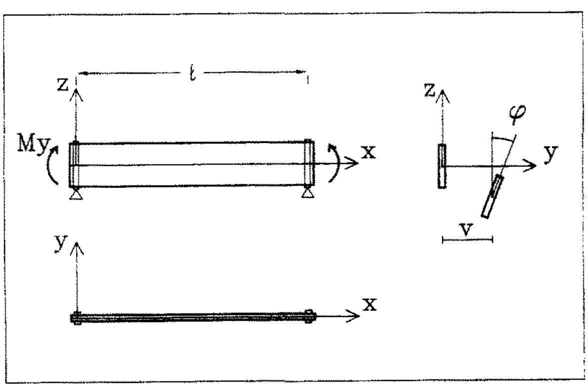

Fig. 1. Problem geometry and reference system.

σ_{xx} in terms of ε_{xx} only by means of the elastic modulus E. This can also be done in the viscous case by introducing an analogous function that will be denoted by $E(t)$. If $\varepsilon_{xx}(t) = 0$ for $t < 0$, the axial stress σ_{xx} can be expressed in the following form (Christensen, 1971; Leitman and Fisher, 1973):

$$\sigma_{xx}(t) = E(0)\varepsilon_{xx}(t) + \int_0^t \dot{E}(t-\tau)\varepsilon_{xx}(\tau)\,\mathrm{d}\tau = E_0\big(\varepsilon_{xx}(t) + e(t) * \varepsilon_{xx}(t)\big), \quad (1)$$

where $E_0 = E(0) > 0$, $e(t) = \dot{E}(t)/E_0$, $*$ denotes convolution and it is assumed that $E(t)$ is an absolutely continuous function on $[0, \infty)$. It is then useful to introduce the function $G(t)$ which gives the shear stresses by means of a quantity analogous to the shear elastic modulus G, often used in the elastic torsion problem. The following laws therefore hold:

$$\sigma_{xy}(t) = G(0)\varepsilon_{xy}(t) + \int_0^t \dot{G}(t-\tau)\varepsilon_{xy}(\tau)\,\mathrm{d}\tau = G_0\big(\varepsilon_{xy}(t) + g(t) * \varepsilon_{xy}(t)\big), \quad (2)$$

$$\sigma_{xz}(t) = G(0)\varepsilon_{xz}(t) + \int_0^t \dot{G}(t-\tau)\varepsilon_{xz}(\tau)\,\mathrm{d}\tau = G_0\big(\varepsilon_{xz}(t) + g(t) * \varepsilon_{xz}(t)\big), \quad (3)$$

where $G_0 = G(0) > 0, g(t) = \dot{G}(t)/G_0$, and it is assumed that $E(t)$ is an absolutely continuous function on $[0, \infty)$.

2.2. *Equilibrium equations*

The balance equations ruling the problem will now be formulated assuming that:

- —the beam has a constant and doubly symmetric cross-section with centroid on the x-axis;
- —the beam is pinned and torsionally clamped at the ends;
- —the beam undergoes two bending moments M_y at the ends for $t \geqslant 0$ and a permanent disturbance due to the imperfections $\phi_0(x)$;
- —the assumption of small deformation holds;
- —the inertia forces and the non-uniform torsion are neglected.

The function $\phi_0(x)$ describes the rotation distribution existing before the load application, which is due to imperfection. It should be noted that the choice of the disturbance was an arbitrary one but a different type of disturbance may be introduced by means of an imperfection function $v_0(x)$ standing for the deviations of the centroid from the x-axis. The solution corresponding to this different case will be presented in Section 6, in which it will be shown that the asymptotic behaviour is the same.

The beam shown in Fig. 1 is now subjected to the external forces from the instant $t = 0$. With the above, the bending and torsional actions for this beam (De Saint Venant theory) can be expressed by means of the displacement and rotation components with equations similar in form to those of the elastic case. Assuming the x-axis placed at the centroid of the cross-section and by denoting the displacement along the y-axis of the points in the axis of the beam as $v(x, t)$, the following relationship holds:

$$M_z(t) = -J_z E_0\left(\frac{\partial^2 v(x,t)}{\partial x^2} + e(t) * \frac{\partial^2 v(x,t)}{\partial x^2}\right), \quad (4)$$

in which J_z denotes the momentum of inertia with respect to the z-axis.

By denoting the rotation of the cross-section by $\phi(x, t)$, the torque moment can be written in the following way:

$$M_t(t) = J_t G_0 \left(\frac{\partial \phi(x,t)}{\partial x} + g(t) * \frac{\partial \phi(x,t)}{\partial x} \right), \tag{5}$$

where J_t is the torsion modulus.

The flexural and torsional balance equations in the local form (Murray, 1986; Timoshenko and Gere, 1961), provide the following system:

$$J_z E_0 \left(\frac{\partial^2 v(x,t)}{\partial x^2} + e(t) * \frac{\partial^2 v(x,t)}{\partial x^2} \right) = M_y(\phi(x,t) + \phi_0(x)), \tag{6a}$$

$$J_t G_0 \left(\frac{\partial^2 \phi(x,t)}{\partial x^2} + g(t) * \frac{\partial^2 \phi(x,t)}{\partial x^2} \right) = M_y \frac{\partial^2 v(x,t)}{\partial x^2}, \tag{6b}$$

and the boundary conditions are as follows:

$$v(0,t) = v(\ell,t) = \phi(0,t) = \phi(\ell,t) = 0, \tag{7}$$

where ℓ is the beam length. By substituting the expression of the second derivative of $v(x,t)$ obtained from the second equation, in the first and dividing by $J_z E_0 J_t G_0$, the following equation in the only unknown function $\phi(x,t)$ can be obtained:

$$\frac{\partial^2 \phi(x,t)}{\partial x^2} + e(t) * \frac{\partial^2 \phi(x,t)}{\partial x^2} + g(t) * \frac{\partial^2 \phi(x,t)}{\partial x^2} + e(t) * g(t) * \frac{\partial^2 \phi(x,t)}{\partial x^2} + \frac{M_y^2}{E_0 G_0 J_z J_t} \phi(x,t) = -\frac{M_y^2}{E_0 G_0 J_z J_t} \phi_0(x). \tag{8}$$

3. SOLUTION OF THE PROBLEM

Since the orthogonal series $\sin(n\pi x/\ell)$ is complete in the space of the functions satisfying the boundary conditions, the problem unknown can be assumed in the following form:

$$\phi(x,t) = \sum_{n=1}^{\infty} \phi_n(t) \sin \frac{n\pi x}{\ell} \tag{9}$$

and the function describing the imperfections in the form (by Fourier expansion):

$$\phi_0(x) = \sum_{n=1}^{\infty} \phi_{0n} \sin \frac{n\pi x}{\ell}. \tag{10}$$

Consequently, the following equation must be satisfied for every n:

$$-(\phi_n(t) + e(t) * \phi_n(t) + g(t) * \phi_n(t) + e(t) * g(t) * \phi_n(t)) + m_n^2 \phi_n(t) = -m_n^2 \phi_{0n}, \tag{11}$$

where m_n represents the ratio between the bending moment M_y and the nth Euler buckling moment

$$m_n = \frac{M_y \ell}{\sqrt{E_0 G_0 J_z J_t}\, n\pi}. \tag{12}$$

Therefore, the Fourier coefficient $\phi_n(t)$ is the solution of the second type Volterra equation (11). This solution, that exists and is unique for every closed interval $[0, t]$ with $t < \infty$, can easily be obtained by means of the Laplace transforms. Using the convolution theorem the following expression is obtained:

$$-(1+\bar{e}(s))(1+\bar{g}(s))\bar{\phi}_n(s)+m_n^2\bar{\phi}_n(s) = -\frac{m_n^2\phi_{0n}}{s}, \tag{13}$$

so that

$$\bar{\phi}_n(s) = \frac{m_n^2\phi_{0n}}{s}\frac{1}{(1+\bar{e}(s))(1+\bar{g}(s))-m_n^2}. \tag{14}$$

The inverse transform of $\bar{\phi}_n(s)$ now needs to be determined. The difficulty of this operation essentially depends on the expression of $e(t)$ and $g(t)$; in general, it can be carried out by approximate procedures already known in literature, as described in Christensen (1971).

The displacements $v(x, t)$ can be subsequently obtained from the rotations by means of the equilibrium equation. By expanding the function $v(x, t)$ into sine series and introducing the relation of eqn (6b), the following relation between the Fourier coefficient of $v(x, t)$ and $\phi(x, t)$ can be obtained:

$$J_t G_0(\phi_n(t)+g(t)*\phi_n(t)) = M_y v_n(t). \tag{15}$$

This equation makes it possible to write the lateral displacement in the form:

$$v(x,t) = \frac{J_t}{M_y}\sum_{n=1}^{\infty}(\phi_n(t)+g(t)*\phi_n(t))\sin\frac{n\pi x}{\ell}. \tag{16}$$

4. STABILITY DISCUSSION

In this section the asymptotic behaviour of the beam ($t \to \infty$) loaded by a moment smaller than the Euler moment ($m_n < 1$) is analysed. The beam is defined as stable if the displacements are bound and tend to a finite limit. Some characteristics of the solution can be deduced from the properties of the Laplace transform and, in particular, the terms $\phi_n(t)$ are bound for $t \to \infty$ if their transforms $\bar{\phi}_n(s)$ satisfy the following conditions:

$$\lim_{s\to 0} s\bar{\phi}_n(s) < \infty, \tag{17}$$

$$\exists\, \bar{\phi}_n(s) \quad \forall s > 0. \tag{18}$$

Here, the weak fading memory condition introduced by Fabrizio and Lazzari (1991) is assumed, i.e.

$$\int_0^\infty |e(t)|\,\mathrm{d}t < \infty, \quad \int_0^\infty |g(t)|\,\mathrm{d}t < \infty, \tag{19}$$

and the additional property holds:

$$\int_0^\infty e(t)\,\mathrm{d}t > -1, \quad \int_0^\infty g(t)\,\mathrm{d}t > -1. \tag{20}$$

This last property ensures that the material is solid because $E(\infty)$ and $G(\infty)$ exist and are positive. Therefore the transforms $\bar{e}(s)$ and $\bar{g}(s)$ exist for $s \geqslant 0$. It can now be concluded that the condition (17) is satisfied and

$$\lim_{s\to 0} s\bar{\phi}_n(s) = \frac{\phi_{0n}m_n^2}{(1+\bar{e}(0))(1+\bar{g}(0))-m_n^2}. \tag{21}$$

It is also assumed that the relaxation functions $E(t)$ and $G(t)$ are compatible with thermodynamics, i.e. the following conditions, determined by Fabrizio and Morro (1988), hold:

$$\int_0^\infty e(t)\sin\omega t\,\mathrm{d}t < 0 \quad \forall\omega > 0; \quad \int_0^\infty g(t)\sin\omega t\,\mathrm{d}t < 0 \quad \forall\omega > 0. \tag{22}$$

The previous properties (19) and (22) allow us to affirm that $(1+\bar{e}(s))$ and $(1+\bar{g}(s))$ are positive for every $s \geqslant 0$, as proved by Fabrizio and Lazzari (1991) and Giorgi (1989), and, observing eqn (14), it can be concluded that the condition (18) is satisfied for all terms if

$$m_1 < \min_{s\in[0,\infty)}\left(\sqrt{(1+\bar{e}(s))(1+\bar{g}(s))}\right). \tag{23}$$

Consequently, some bending moment values for which the solution is stable always exist for a viscoelastic solid. It must be pointed out that (19, 20, 22) are the same assumptions which ensure the existence and uniqueness of the classic viscoelastic problem (Fabrizio and Lazzari, 1991).

The previous stability condition is very general because it is based only on the assumption of solid material and thermodynamic restrictions. More interesting and useful properties of the solution can be obtained by introducing the further assumption (monotonicity):

$$e(t) \leqslant 0, \quad g(t) \leqslant 0, \tag{24}$$

which is not due to thermodynamic restrictions but has been verified by all the relaxation functions deduced from experimental observations (Fabrizio and Morro, 1988). In this case $\bar{e}(0) \leqslant \bar{e}(s)$ and $\bar{g}(0) \leqslant \bar{g}(s)$ and the stability condition becomes

$$m_1 < m_v^{cr} = \sqrt{(1+\bar{e}(0))(1+\bar{g}(0))}, \tag{25}$$

where m_v^{cr} is the critical ratio between the moment M_y and the Euler moment; the viscoelastic critical moment assumes the form:

$$M_{yv}^{cr} = \frac{\pi}{\ell}\sqrt{J_z E_0(1+\bar{e}(0))J_t G_0(1+\bar{g}(0))}. \tag{26}$$

The previous expression has a remarkable physical significance because $E_0(1+\bar{e}(0)) = E(\infty)$ and $G_0(1+\bar{g}(0)) = G(\infty)$. Therefore the viscous critical moment is equal to the Eulerian moment evaluated with reference to the equilibrium moduli $E(\infty)$ and $G(\infty)$. As expected, the critical viscous load is lower than the Euler load and depends on both the functions $E(t)$ and $G(t)$ describing the material behaviour under multi-axial strain. It must also be noted that m_v^{cr} does not depend on the entity of the disturbance and, more generally, on the known terms of the balance equation.

The closed-form solution cannot be performed in general terms, however some aspects of the evolution in time of the internal actions can still be posed in evidence. In particular, it can be observed that the lateral displacements $v(x,t)$ progress in time more slowly than the rotations. This fact can be shown by writing the ratio $v_n(t)/v_n(0)$ between the displacement at a generic instant and at the initial instant, for a generic nth component, and by comparing this with the corresponding ratio $\phi_n(t)/\phi_n(0)$ between the rotations. In fact, the following relationship can be derived from eqns (16) and (24):

$$\left|\frac{v_n(t)}{v_n(0)}\right| = \left|\frac{\phi_n(t)}{\phi_n(0)} + \frac{g(t)*\phi_n(t)}{\phi_n(0)}\right| < \left|\frac{\phi_n(t)}{\phi_n(0)}\right|. \tag{27}$$

Taking into account that the bending and torque moments are related to $v(x,t)$ and $\phi(x,t)$ respectively by means of two linear operators [eqns (4), (5)] it can be deduced that the torque moment also evolves in time more rapidly than the bending moment. Consequently the torsion action becomes more important in time for evaluating reliability.

5. SOLUTION FOR THE THREE-ELEMENT MODEL

When the viscous model is particularly simple, an analytical solution can be achieved. At this point, the three-element model schematized in Fig. 2, is considered. This case represents the simplest spring-dashpot model that can simulate the behaviour of a linear viscoelastic solid material and permits the carrying out of some significant observations.

The model consists of an elastic spring in series with a Kelvin element (spring and dashpot in parallel). The viscous kernels have the following analytical expressions [reported in Flugge (1975)]:

$$e(t) = -E^* \exp(-\eta t), \tag{28a}$$

$$g(t) = -G^* \exp(-\gamma t), \tag{28b}$$

where the constants are positive and are given by

$$E^* = \frac{E_0}{\mu_E}, \quad \eta = \frac{E_0+E_1}{\mu_E}, \quad G^* = \frac{G_0}{\mu_G}, \quad \gamma = \frac{G_0+G_1}{\mu_G}. \tag{29}$$

The Laplace transforms can be easily determined in the following form:

$$\bar{e}(s) = -\frac{E^*}{s+\eta}, \quad \bar{g}(s) = -\frac{G^*}{s+\gamma}, \tag{30}$$

and introduced into eqn (14) they provide the following:

$$\bar{\phi}_n(s) = \frac{m_n^2 \phi_{0n}}{s} \frac{(s+\eta)(s+\gamma)}{a_n s^2 + b_n s + c_n}, \tag{31}$$

where the introduced constants a_n, b_n, c_n are equal to

$$a_n = 1 - m_n^2, \tag{32a}$$

$$b_n = (\eta+\gamma)(1-m_n^2) - (E^*+G^*), \tag{32b}$$

$$c_n = \eta\gamma(1-m_n^2) - E^*\gamma - G^*\eta + E^*G^*. \tag{32c}$$

The inversion transform of a ratio between polynomials of this type can be found by means of the roots of the denominator that, in this case, are three and are given by

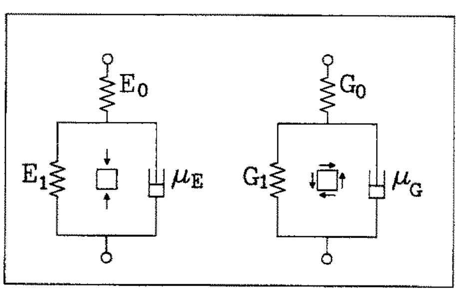

Fig. 2. Three-element models.

$$s_{0n} = 0, \tag{33a}$$

$$s_{1n} = \frac{-b_n + \sqrt{b_n^2 - 4a_n c_n}}{2a_n}, \tag{33b}$$

$$s_{2n} = \frac{-b_n - \sqrt{b_n^2 - 4a_n c_n}}{2a_n}. \tag{33c}$$

When the bending moment is lower than the critical Euler moment, the term $b_n^2 - 4a_n c_n$ is positive, so that the roots s_{1n} and s_{2n} are always real. The inverse transform of the nth Fourier coefficient can be carried out and written in the form:

$$\phi_n(t) = m_n^2 \phi_{0n} \left[\frac{\eta\gamma}{c_n} + \sum_{\alpha=1}^{2} \frac{(s_{\alpha n} + \eta)(s_{\alpha n} + \gamma)}{3a_n s_{\alpha n}^2 + 2b_n s_{\alpha n} + c_n} \exp(s_{\alpha n} t) \right]. \tag{34}$$

Now, the component $v_n(t)$ can be obtained from eqn (15). It should be observed that each displacement component varies in time with a different law and the viscous deformation is not proportional to initial deformation.

The viscous critical load can be easily evaluated by means of eqn (25) and the following form is obtained:

$$m_v^{cr} = \sqrt{\left(1 - \frac{E^*}{\eta}\right)\left(1 - \frac{G^*}{\gamma}\right)}. \tag{35}$$

Many authors have investigated the behaviour of viscoelastic columns describing the material by a three-element model. An interesting comparison between that problem and the problem analysed here can be made in the particular case in which the Poisson ratio v is a constant during the viscous phenomena. In this case only one viscous function is required to describe the constitutive law mapping the strain tensor into the stress tensor. In the considered case, the function $E(t)$ can be assumed to be independent and the ratio between $G(t)$ and $E(t)$ can be imposed so that it does not vary in time and maintains the initial value between $G(0)$ and $E(0)$, i.e. $G(t) = E(t)/[2(1+v)]$. For the three-element model this condition provides the two relationships $\gamma = \eta$ and $G^* = E^*$. Thus the viscous critical load can be written in the following form:

$$m_v^{cr} = 1 - \frac{E^*}{\eta}. \tag{36}$$

The authors who have studied the problem of viscoelastic columns obtained the following ratio p_v^{cr} between the viscous critical load and the Euler load for a three-element model (Dost and Glocker, 1985; Szyszkowski and Glockner, 1985; Russo Spena and Sparacio, 1989):

$$p_v^{cr} = 1 - \frac{E^*}{\eta}. \tag{37}$$

This value is exactly equal to that obtained in the problem of lateral stability of a beam in the special case of v constant in time. However, even if the asymptotic critical load is the same, the two problems are different and both the rotations and the lateral displacements of the cross-section evolve in time through a different law with respect to the displacements of the column.

6. SOLUTION IN THE CASE OF THE IMPERFECTION $v_0(x)$

In this chapter, the behaviour of a beam subjected to imperfections given by the function $v_0(x)$ is analysed. The balance equations, corresponding to eqns (6a–b) have the following form:

$$J_z E_0 \left(\frac{\partial^2 v(x,t)}{\partial x^2} + e(t) * \frac{\partial^2 v(x,t)}{\partial x^2} \right) = M_y \phi(x,t), \tag{38a}$$

$$J_t G_0 \left(\frac{\partial^2 \phi(x,t)}{\partial x^2} + g(t) * \frac{\partial^2 \phi(x,t)}{\partial x^2} \right) = M_y \left(\frac{\partial^2 v(x,t)}{\partial x^2} + \frac{\partial^2 v_0(x)}{\partial x^2} \right) \tag{38b}$$

and the following equation in the sole unknown $\phi(x,t)$, corresponding to eqn (8), is obtained:

$$\frac{\partial^2 \phi(x,t)}{\partial x^2} + e(t) * \frac{\partial^2 \phi(x,t)}{\partial x^2} + g(t) * \frac{\partial^2 \phi(x,t)}{\partial x^2} + e(t) * g(t) * \frac{\partial^2 \phi(x,t)}{\partial x^2} + \frac{M_y^2}{E_0 J_z G_0 J_t} \phi(x,t)$$
$$= \frac{M_y}{G_0 J_t} \left(\frac{\partial^2 v_0(x)}{\partial x^2} + e(t) * \frac{\partial^2 v_0(x)}{\partial x^2} \right). \tag{39}$$

Unlike the previous case, it can be noted that the known term is not a constant but varies in time. The solution can be obtained by following the same process used for the $\phi_0(x)$ type imperfections, to attain the following series of equations with respect to the component $\phi_n(t)$ of the unknown function expanded in sine series:

$$-(\phi_n(t) + e(t) * \phi_n(t) + g(t) * \phi_n(t) + e(t) * g(t) * \phi_n(t)) + m_n^2 \phi_n(t) = -\tilde{v}_{0n} - e(t) * \tilde{v}_{0n}, \tag{40}$$

where m_n is defined in eqn (12) and

$$\tilde{v}_{0n} = \frac{M_y}{J_t G_0} v_{0n}, \tag{41}$$

where v_{0n} is the nth component of $v_0(x)$. The Laplace transform of the nth component assumes the form:

$$\bar{\phi}_n(s) = \frac{\tilde{v}_{0n}(1 + s\bar{e}(s))}{s} \frac{1}{(1 + \bar{e}(s))(1 + \bar{g}(s)) - m_n^2} \tag{42}$$

and, for the three-element model, the following inverse transform is obtained:

$$\phi_n(t) = \tilde{v}_{0n} \left[\frac{\eta\gamma}{c_n} + \sum_{\alpha=1}^{2} \frac{(s_{\alpha n} + \eta - s_{\alpha n} E^*)(s_{\alpha n} + \gamma)}{3a_n s_{\alpha n}^2 + 2b_n s_{\alpha n} + c_n} \exp(s_{\alpha n} t) \right], \tag{43}$$

where the constants a_n, b_n, c_n are defined in eqns (32a–c) and the roots $s_{\alpha n}$ ($\alpha = 1, 2$) in eqns (33a–c). Consequently, the stability conditions coincide with those of the case examined in the previous chapters, even if the evolution in time differs. In fact it can be noted that, for a three-element model, the ratio $\phi_n(t)/\phi_n(0)$ asymptotically tends to a different value of the previous case and consequently the rotations undergo a different gain with respect to the previous case when $t \to \infty$.

7. APPLICATION

A numerical example can be developed assuming a three-element model for which $E_0 = E_1 = 30{,}000$ MPa and $\mu_E = 1.0 \times 10^6$ MPa days. Such a model can describe, fairly accurately, the behaviour of concrete undergoing a uni-axial stress for period of up to 300 days. For this material, in usual environmental conditions, the Poisson ratio gradually drops from $\nu = 0.2$ to almost zero. Assuming $\eta = \gamma$, the constants G_0 and G_1 may be assigned in such a way as to satisfy the relation $G(0) = E(0)/[2(1+0.2)]$ due to the condition $\nu = 0.2$ when $t = 0$ and the relation $G(\infty) = E(\infty)/2$ due to the condition $\nu = 0$ when $t \rightarrow \infty$. Therefore the values $G_0 = 12{,}500$ MPa and $G_1 = 18{,}750$ MPa are fixed for the constants describing shear deformability.

In this case the ratio $m_v^{cr} = 0.548$ between the viscous critical load and the Euler load is obtained. It follows that the reliability of stability reduces considerably in time.

A sinusoidal disturbance $\phi_0(x) = \phi_0 \sin(\pi x/\ell)$ is considered, so the displacement components with index $n > 1$ are all zero. Figure 3(a) shows the ratios $\phi_1(t)/\phi_1(0)$ between the maximum rotation at the instant t and the maximum rotation at the initial instant for different values of m_1. The dotted line shows this ratio in the case of the different disturbance $v_0(x) = v_0 \sin(\pi x/\ell)$ and $m_1 = 0.5$. In this case, the rotation increases more rapidly although it tends to the same asymptotic value. Figure 3(b) reports the ratios $v_1(t)/v_1(0)$ for different values of the load m_1 and $\phi_0(x)$ type imperfections. By comparing Fig. 3(a) and Fig. 3(b), it can be noted that, as expected, the lateral displacements are amplified less than the rotations.

At this point it is interesting to analyse the evolution of bending and torsion actions. As can be seen in Figs 4(a)–(b), the ratio between the action at the instant t and that at the

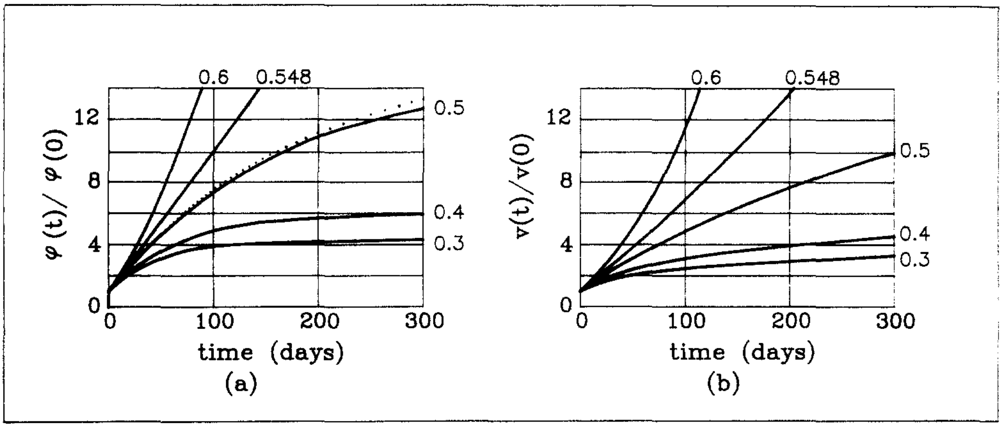

Fig. 3. Displacements in time: (a) rotations; (b) lateral displacements.

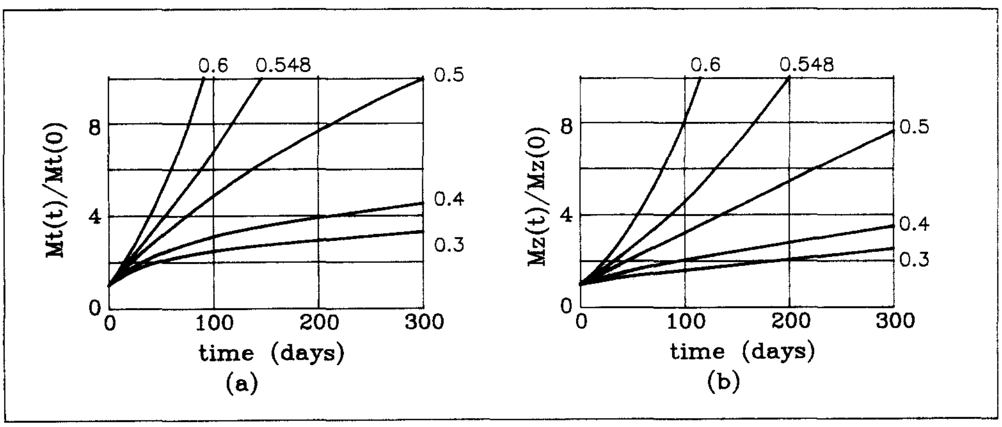

Fig. 4. Internal actions in time: (a) torque moment; (b) bending moment.

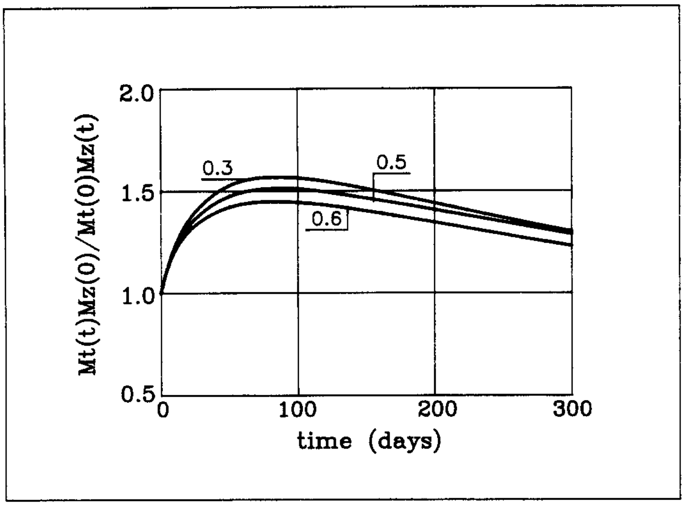

Fig. 5. Ratio $M_t(t)M_z(0)/M_t(0)M_z(t)$ in time.

initial instant increases considerably in each case even when m_l is notably less than the critical load. The ratio $M_t(t)M_z(0)/M_t(0)M_z(t)$ between the amplification of the torque moment and the amplification of the bending moment for different values of m_l is presented in Fig. 5. It is evident that the growth of the torque moment is considerably faster than that of the bending moment both in the case of stable beams, in which the moments tend to a finite value, and in the case of unstable beams.

8. CONCLUSION

In this work the influence of the viscosity on the lateral stability of viscoelastic imperfect beams has been analysed by means of a quasi-static approach, obtaining a solution through series expansion and Laplace transforms.

By assuming a solid material compatible with thermodynamics, it is possible to relate the asymptotic behaviour to the bending moment and to the viscoelastic parameters of the material. Consequently a stability condition and a viscous critical moment are established. This critical moment is smaller than the Euler moment and depends neither on the entity nor on the type (rotations or displacements) of the imperfection. Furthermore a qualitative analysis of the solution has shown that the rotation and the torque moment grow in time faster than the displacement and the bending moment, respectively.

A closed form solution can be performed for a three-element model. In the case of the Poisson ratio being constant in time an interesting observation can be made: the ratio between the viscous critical moment and the Euler moment is equal to the ratio between the viscous critical load and the Euler load established for an imperfect column.

A numerical example has been developed with reference to a concrete beam. This application shows the main aspects of the problem and demonstrates the viscosity notably affects reliability in real cases.

REFERENCES

Christensen, R. M. (1971). *Theory of Viscoelasticity—An Introduction.* Academic Press, New York.

Dost, S. and Glockner, P. G. (1985). The stability of viscoelastic perfect columns: a dynamic approach. *Int. J. Solids Structures* **21**(6), 545–559.

Fabrizio, M. and Lazzari, B. (1991). On the existence and the asymptotic stability of solutions for linearly viscoelastic solids. *Arch. Rat. Mech. Anal.* **116,** 139–152.

Fabrizio, M. and Morro, A. (1988). Viscoelastic relaxation functions compatible with thermodynamics. *J. Elasticity* **19,** 63–75.

Flugge, W. (1975). *Viscoelasticity* (2nd Edn). Springer, Berlin.

Giorgi, C. (1989). Alcune conseguenze delle restrizioni termodinamiche per mezzi viscoelstici lineari. *Quad. Dip. Mat. Univ. Cattolica S. Cuore,* **6.**

Hilton, H. H. (1961). On the nonexistence of finite critical times for generalized linear viscoelastic columns with arbitrary initial curvature. *J. Aeronaut. Sci.* **28**, 655–656.

Kempner, J. and Pohle, F. V. (1953). On the nonexistence of finite critical time for linear viscoelastic columns. *J. Aeronaut. Sci.* **20**, 572–573.

Leitman, J. M. and Fisher, G. M. C. (1973). The linear theory of viscoelasticity. In *Handbuch der Physic* (Edited by C. Truesdell). Vol. VIa/3, pp. 10–31. Springer, Berlin.

Murray, N. W. (1986). *Introduction to the Theory of Thin Walled Structures*. Oxford Science Publications, Oxford.

Russo Spena, F. and Sparacio, R. (1989). Stabilita' asintotica di travi in regime di elasto-viscosita' ereditaria. Proc. Symp. "Convegno Nazionale in ricordo di Riccardo Baldacci e Michele Capurso", Rome, 25–27 October, 245–254.

Szyszkowski, S. and Glockner, P. G. (1985). The imperfect linearly viscoelastic column. *Int. J. Engng Sci.* **23**(10), 1113–1120.

Timoshenko, S. P. and Gere, J. M. (1961). *Theory of Elastic Stability*. McGraw-Hill, New York.